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THE STARLIKE TREES WHICH SPAN A HYPERCUBE

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Abstract—A connected bipartite graph is called equitable if it has the same number of nodes in each of its two colors. A starlike tree with b branches is a subdivision of the star $K_{1,b}$ with $b \geqslant 3$. We prove that a starlike tree T with b branches, where $3 \leqslant b \leqslant n$, having 2^n nodes spans the hypercube Q_n if and only if T is equitable. This extends a result of Nebesky who had made this observation for $b = 3$.

1. EQUITABLE STARLIKE TREES

There is much interest in the theoretical study of the structure of hypercubes because of their successful utilization as the underlying architecture of massively parallel computers.

The hypercube Q_n is recursively defined using cartesian products [1, p. 23] as follows: $Q_1 = K_2$ and $Q_{n+1} = Q_n \times K_2$. A *starlike tree* is a tree with just one node of degree greater than two. This node is called the *junction*, and its degree is the number of *branches*. Starlike trees are, of course, bipartite as are all trees and hence bicolorable. We adopt the colors black and white, with *the junction always a black node*. If there are as many black nodes as white nodes, the starlike tree is called *equitable*, as is any connected bigraph.

It is known, Nebesky [2], that equitable starlike trees with 2^n nodes span Q_n if the degree of the junction is 3. We extend this by showing that the degree of the junction may be any integer between 3 and n. First, we will require several lemmas. We omit the trivial proof of the first of these.

Lemma 1

A starlike tree is equitable if and only if it has exactly one white endnode. □

Henceforth, in view of Lemma 1, we only consider an equitable k-branched starlike tree denoted $S(a, b_1, b_2, \ldots, b_{k-1})$ for which a is an odd integer, the b_i are even integers, and $b_1 = \max b_i$. The unique branch A with a white endnode has a nodes other than the junction. The other branches will be denoted $B_1, B_2, \ldots, B_{k-1}$. Clearly $|V(S)| = 1 + a + \Sigma b_i$.

Lemma 2

Coloring the nodes of Q_n black and white, given a black node x and a white node y of Q_n, there exists a hamiltonian x–y path.

Proof. As this is trivial for $n = 1$, assuming its truth up to n, we prove it for $n + 1$. Let x be a black and y a white node of Q_{n+1}. Now Q_{n+1} may be considered as in [3] as two copies of Q_n, called Q_n^L (Q_n lower) and Q_n^U (Q_n upper), with 2^n "vertical" edges joining corresponding nodes in both copies. Furthermore, the cutset of 2^n "vertical" edges may be selected in $n + 1$ ways. Now, given two nodes of Q_{n+1}, it is always possible to find an edge cutset so that the two nodes belong to node-disjoint copies of Q_n.

Without loss of generality, therefore, let $x \in V(Q_n^L)$ and $y \in V(Q_n^U)$.

Choose a white node w in Q_n^L with corresponding black node w' in Q_n^U. By the inductive hypothesis, there exist hamiltonian paths x–w and w'–y in Q_n^L and Q_n^U, respectively. Then an x–y hamiltonian path for Q_{n+1} is $(x\text{-}w) \cup (ww') \cup (w' - y)$, and the lemma is proven. □

The next result was found and proved by Nebesky [2]. We rediscovered it independently as part of our proof of Theorem 1 before learning of Nebesky's work by serendipity.

Theorem A

For $n \geqslant 3$, equitable 3-branched starlike trees on 2^n nodes span Q_n. □

Lemma 3

Given a set of nodes $\{v_1, v_2, \ldots, v_s\}$ of the same color in Q_n, and a set of even positive integers $\{e_1, e_2, \ldots, e_s\}$ such that $\Sigma e_i = 2^n$, then there exist node-disjoint paths $L_1, L_2, \ldots, L_s$ containing $e_1, e_2, \ldots, e_s$ nodes, respectively, such that each v_i is an endnode of L_i, and the L_i together span Q_n.

Proof. As the lemma is trivial for $n = 2$, we assume its truth up to the integer n, and verify it for $n + 1$. Let $v_1, \ldots, v_s$ be a set of black nodes in Q_{n+1} and let $e_1, \ldots, e_s$ be a set of even positive integers whose sum is 2^{n+1}. There are two cases.

Case 1

There is a division of Q_{n+1} into Q_n^L and Q_n^U such that all the v_i belong to Q_n^L.

Now select even positive integers $d_1, d_2, \ldots, d_s$ whose sum is 2^n with $d_i \leqslant e_i$. By the inductive hypothesis, there exist node-disjoint paths $H_1, \ldots, H_s$ together spanning Q_n^L, such that each H_i contains d_i nodes and has v_i as an endnode.

We partition $\{1, \ldots, s\}$ into $C = \{i: d_i = e_i\}$ and $D = \{i: d_i < e_i\}$. For $i \in D$, let the endnode of H_i other than v_i be x_i. Clearly the x_i are white nodes in Q_n^L. Let their corresponding nodes in Q_n^U be w_i, so the w_i are black. Also $\Sigma d_i = 2^n$ implies

$$\sum_{i \in D} (e_i - d_i) = 2^n.$$

Then by the inductive hypothesis, there exist node-disjoint paths J_i, $i \in D$, which together span Q_n^U, such that J_i was w_i as an endnode and contains $e_i - d_i$ nodes. Now we define the subgraphs

$$L_i = \begin{cases} H_i, & i \in C, \\ H_i \cup x_i w_i \cup J_i, & i \in D. \end{cases}$$

Each L_i has e_i nodes. The L_i are node-disjoint and together span Q_{n+1}, and v_i is an endnode of L_i.

Case 2

There is no division such that all the v_i belong to Q_n^L.

Assume, with out loss of generality, that $v_1, \ldots, v_k$ lie in Q_n^L, and $v_{k+1}, \ldots, v_s$ lie in Q_n^U. Let $f_1 = e_1 + e_2 + \cdots + e_k$ and let $f_2 = e_{k+1} + \cdots + e_s$. Clearly $f_1 + f_2 = 2^{n+1}$. We require two subcases.

Case 2(a). $f_1 = f_2 = 2^n$.

By the inductive hypothesis, there exist node-disjoint paths $H_1, \ldots, H_s$, such that v_i is an endnode of H_i and H_i contains e_i nodes for $i = 1, \ldots, s$. Also $H_1, \ldots, H_k$ together span Q_n^L and $H_{k+1}, \ldots, H_s$ together span Q_n^U. Then clearly the H_i together span Q_{n+1}, and the lemma is proven in this case.

Case2(b). $f_1 > s^n$ (without loss of generality).

Select a set of even positive integers $d_1, \ldots, d_k$ with $d_i \leqslant e_i$ and $d_1 + \cdots + d_k = 2^n$. By the inductive hypothesis, there exist node-disjoint paths $H_1, \ldots, H_k$ with endnodes $v_1, \ldots, v_k$ such that H_i Has d_i nodes, and together the H_i span Q_n^L.

Again we partition $\{1, \ldots, k\}$ into $C = \{i: d_i = e_i\}$ and $D = \{i: d_i < e_i\}$. For $i \in D$, let the endnode of H_i other than v_i be x_i, and let its corresponding node in Q_n^U be w_i. The w_i are black. Applying the set of nodes $\{w_i: i \in D\} \cup \{v_i: k + 1 \leqslant i \leqslant s\}$ and the set of even positive integers $\{e_i - d_i: i \in k + 1 \leqslant i \leqslant s\}$ to the inductive hypothesis, we infer the existence of node-disjoint paths J_i which together span Q_n^U, such that for $i \in D$, w_i is an endnode of J_i, J_i has $e_i - d_i$ nodes, and for $i = k + 1, \ldots, s$, v_i is an endnode of J_i, and J_i has e_i nodes. Defining

$$L_i = \begin{cases} H_i, & i \in C, \\ H_i \cup x_i w_i \cup J_i, & i \in D, \\ J_i, & i = k + 1, \ldots, s, \end{cases}$$

the L_i satisfy the lemma for Case 2(b). □

2. HYPERCUBE-SPANNING STARLIKE TREES

We state and prove our main result.

Theorem 1

Every equitable k-branched starlike tree on 2^n nodes with $3 \leqslant k \leqslant n$ spans Q_n.

Proof. The theorem holds for $k = 3$ by Theorem A. We assume it is true up to k, and verify it for $k + 1$. Let $G = S(a, b_1, b_2, \ldots, b_k)$ be a (k + 1)-branched equitable starlike tree on 2^{n+1} nodes, where $k \leqslant n$. Denote the branches by $A, B_1, \ldots, B_k$.

Case 1 ($a + b_1 > 2^n$)

Denote the junction node by j, and let u be the node of B_1 adjacent to j. Add an edge e incident to the endnodes of A and B_1 thereby creating a unicyclic graph $G + e$ whose unique cycle we call C. Let L be the path in C on 2^n nodes with endnode u, not including j. Let the other endnode be w. Let x be the node of C adjacent to w and not in L. Now $G + e - L$ is clearly an equitable k-branched starlike tree on 2^n nodes which can, by the inductive hypothesis, be embedded in Q_n^L. Now let ju and xw be "vertical" edges so that u and w are nodes of Q_n^U of opposite color.

Then by Lemma 2, there exists a u-w hamiltonian path for Q_n^U which we take to be L, thereby embedding $G + e$, and hence G, *in* Q_{n+1}.

Case 2 ($a + b_1 < 2^n$)

We will show that by possibly relabeling the branches B_2 through B_k, there exists a set of white nodes $u_1, \ldots, u_s$, ($s \leqslant k$), such that $u_i \in B_i$ for $i = 1, \ldots, s$, u_i is adjacent to j if and only if $i = 1$, and if d_i is the number of nodes on the path joining u_i and the endcode of B_i, then $d_1 + \cdots + d_s = 2^n$. If this can be accomplished, then the embedding can be done as follows.

For $i = 1, \ldots, s$, let H_i be the path on B_i with one endnode u_i, and with its other endnode being the endnode of B_i. Then H_i has d_i nodes. Let x_i be the node adjacent to u_i which doesn't belong to H_i.

Now $G - \cup H_i$ is a k-branched equitable starlike tree on 2^n nodes which, by the inductive hypothesis, can be embedded in Q_n^L. The edges ju_1 and $x_i u_i$ for $i = 2, \ldots, s$ are then "vertical" edges, with nodes u_i white and in Q_n^U. By Lemma 3, the H_i together span Q_n^U, so that G is embedded in Q_{n+1}.

We now show that the selection of the u_i under the conditions stated above is possible. It suffices to show that

$$b_1 + \sum_{i=2}^{k} (b_i - 2) \geqslant 2^n. \tag{1}$$

We assume the worst case, i.e. that $k = n$, so expression (1) becomes

$$b_1 + \sum_{i=2}^{n} (b_i - 2) \geqslant 2^n. \tag{2}$$

When $n = 2$, the only equitable 3-branched starlike trees are $S(3, 2, 2)$ and $S(1, 4, 2)$, neither of which involve Case 4. When $n = 3$, we have for Case 4 only $S(3, 4, 4, 4)$, $S(1, 6, 6, 2)$ and $S(1, 6, 4, 4)$ which satisfy expression (2). The cases $n = 4$, 5, 6 are handled similarly and the details are omitted. We present a general argument for $n \geqslant 7$.

After a bit of simplification, expression (2) becomes

$$\sum_{i=1}^{n} b_i \geqslant 2^n + 2n - 2$$

which, upon recalling that

$$\sum_{i=1}^{n} b_i + 1 + a = 2^{n+1},$$

becomes

$$2^n - 2n + 1 \geqslant a. \tag{3}$$

We prove expression (3) by contradiction. Assume that $a > 2^n - 2n + 1$. Since b_1 is the largest of the b_i, it follows that $b_1 \geqslant (2^{n+1} - a - 1)/n$. Then with the assumption that $a > 2^n - 2n + 1$, we have

$$a + b_1 > 2^n - 2n + 1 + (2^{n+1} - a - 1)/n. \tag{4}$$

Being in Case 2, we know that $a + b_1 < 2^n$. We establish a contradiction by showing that $2^n - 2n + 1 + (2^{n+1} - a - 1)/n \geqslant 2^n$, or

$$a \leqslant 2^{n+1} - (2n^2 - n + 1). \tag{5}$$

As $a < 2^n$, we show expression (5) is true by verifying that $2n^2 - n + 1 \leqslant 2^n$, which is clearly true for $n \geqslant 7$, completing the proof. □

Unsolved problems

We have determined precisely which starlike trees span a hypercube. The corresponding problem for caterpillars was studied by Havel and Liebl [3] who found a family of hypercube-spanning caterpillars. It remains to determine a criterion for such spanning caterpillars. More generally, criteria for any tree or unicyclic graph to span a hypercube have not yet been obtained.

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